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Rational interpolation and mixed inverse spectral problem for finite CMV matrices

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Abstract

For finite-dimensional CMV matrices the mixed inverse spectral problem of reconstructing the matrix by its submatrix and a part of its spectrum is considered. A general rational interpolation problem which arises in solving the mixed inverse spectral problem is studied, and the description of the space of its solutions is given. We apply the developed technique to give sufficient conditions for the uniqueness of the solution of the mixed inverse spectral problem.

Keywords: CMV matrices; Verblunsky coefficients; Szegő recurrences; Direct and inverse spectral problems; Spectral measure; Weyl function; Mixed inverse problems

1. Introduction

The theory of CMV matrices, rapidly developing in the recent years ([1–3], see also the expositions in [12–14,9] and the references therein), has a strong background in the theory of Jacobi matrices. The similarity between the Jacobi and CMV matrices not only provides a general concern of investigation, but also permits sometimes to predict the most probable answers. Some of the facts known for the Jacobi matrices can be easily carried over to the CMV case, the others require considerable adjustment. In turn, the spectral theory of the Jacobi matrices is paralleled by

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the spectral theory for the Sturm–Liouville differential operators, which forms another important front of study in this area of research.

Mixed inverse spectral problems (**MISP**) are of special interest in the inverse spectral theory. For this kind of problems one reconstructs a differential or difference operator by a part of its potential and some additional spectral data. Started by Hochstadt and Lieberman [7] for the Sturm–Liouville operators, these problems were extended and refined for many other cases (see [4] for the references). Compared to the "ordinary" inverse spectral problems where the whole potential is to be reconstructed, for MISP one needs to know "less" spectral data.

In what follows, we refer to several key studies of the development of MISP. Let J be an $N \times N$ three-diagonal matrix of the form

$$J = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdot & \cdot & \cdot \\ b_1 & a_2 & b_2 & 0 & \cdot & \cdot & \cdot \\ 0 & b_2 & a_3 & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & b_{N-1} & a_N \end{pmatrix}, \quad a_n, b_n \in \mathbb{R},\ b_n > 0.$$

Consider the a's and b's as a single sequence $a_1, b_1, a_2, b_2, \ldots$, that is,

$$c_{2n-1} = a_n, \qquad c_{2n} = b_n, \quad n \in \mathbb{N}.$$

In [8] Hochstadt proved the discrete version of the Hochstadt–Lieberman theorem:

Theorem. *Let $n \in \mathbb{N}$. Suppose that $c_{N+1}, \ldots, c_{2N-1}$ are known, as well as the eigenvalues $\lambda_1, \ldots, \lambda_N$ of J. Then $c_1, \ldots, c_N$ are uniquely determined.*

It is important to remark that in certain complicated physical systems it is not always possible to know the entire spectrum. A natural question arises if it is possible to reconstruct the Jacobi matrix when we know more than a half of the potential, but less than the whole spectrum. The positive answer is given by Gesztesy and Simon in [5].

Theorem. *Suppose that $1 \le j \le N$ and $c_{j+1}, \ldots, c_{2N-1}$ are known, as well as (any) j of the eigenvalues. Then $c_1, \ldots, c_j$ are uniquely determined.*

The goal of this paper is to study the MISP for CMV matrices. Although the algorithm of solving the problem is similar to that of the Jacobi case, essential difference arises when the uniqueness of the reconstruction is concerned. In the Jacobi case the MISP is reduced to the interpolation of a rational function (specifically, the Weyl function of the unknown submatrix) by its values in the known eigenvalues of the whole matrix. The degree of the rational function (i.e., the sum of the degrees of the numerator and denominator) corresponds to the number of the interpolation points, and both numerator and denominator are monic. So, the uniqueness of the interpolating rational function drops out immediately from the simple fact that a polynomial of degree k with $k+1$ zeros is identically zero, and thus we prove the uniqueness of the reconstructed Jacobi matrix. However, in the CMV case the degree of the interpolating rational function (the Weyl function of the "reduced CMV matrix") is greater by 1 than the number of the interpolation points, and the lacking "piece of information" is given by a restricting condition on the free term of the numerator to be 1. Here the trivial consideration of the Jacobi case fails, and a theory is required of how to find the interpolating rational function with such restriction.

So, the work consists of two parts. In Section 2 we give an approach to the rational interpolation theory adapted for solution of the interpolation problem related to the MISP. The

MISP itself is studied in Section 3. Note that, although in the Jacobi case we always have uniqueness in the MISP (provided the number of lacking entries agrees with the number of the given parameters), in the CMV case a degenerated case is possible where the MISP has infinitely many solutions. The main results of this paper are Theorem 2.12 with a description of the solutions of the rational interpolation problem, and Theorem 3.3 with a sufficient condition for the MISP to have a unique solution.

2. Two-dimensional vector-polynomials and rational interpolation

We start with the following interpolation problem: given points $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and numbers $\omega_1, \omega_2, \ldots, \omega_n \in \overline{\mathbb{C}}$ find a "nice" description of *all* rational functions $P^{(1)}/P^{(2)}$, where $P^{(1)}$ and $P^{(2)}$ are polynomials with complex coefficients, for which

$$\frac{P^{(1)}(z_j)}{P^{(2)}(z_j)} = \omega_j, \quad j = 1, \ldots, n, \tag{2.1}$$

($\omega_j = \infty$ means that the rational function must have a pole at the point z_j). By a "nice" description we understand a description whose form allows us to "control" the degrees of $P^{(j)}(z)$, $j = 1, 2$, and some more parameters such as their leading coefficients and/or the free terms, required in the concrete application of the rational interpolation problem (we will return later on to this topic).

It is advisable to reformulate this problem as a linear problem in the space of two-dimensional polynomial vector-functions (vector-polynomials). Define numbers $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$ by the following rule

$$\begin{cases} \alpha_j^{(1)} := 1, & \alpha_j^{(2)} := -\omega_j, \quad \text{if } \omega_j \neq \infty, \\ \alpha_j^{(1)} := 0, & \alpha_j^{(2)} := 1, \quad \text{if } \omega_j = \infty. \end{cases} \tag{2.2}$$

Then (2.1) implies

$$\alpha_j^{(1)} P^{(1)}(z_j) + \alpha_j^{(2)} P^{(2)}(z_j) = 0, \quad j = 1, \ldots, n, \tag{2.3}$$

where $|\alpha_j^{(1)}| + |\alpha_j^{(2)}| > 0$.

Conversely, if $|P^{(1)}(z_j)| + |P^{(2)}(z_j)| > 0$, $j = 1, \ldots, n$, then (2.3) implies (2.1) with

$$\omega_j := -\frac{\alpha_j^{(2)}}{\alpha_j^{(1)}} \in \overline{\mathbb{C}}, \quad j = 1, \ldots, n. \tag{2.4}$$

However, if $|P^{(1)}(z_j)| + |P^{(2)}(z_j)| = 0$ for some j, then (2.1) does not make sense at the point z_j.

Thus, we reformulated the initial interpolation problem in the following way: find all the pairs of polynomials $P^{(1)}$ and $P^{(2)}$, for which (2.3) holds. The second problem is "almost equivalent" to the first one: if the rational function $P^{(1)}/P^{(2)}$ is a solution of (2.1), then the pair of polynomials $P^{(1)}$ and $P^{(2)}$ is a solution of (2.3) with $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$ defined in (2.2). If $P^{(1)}$ and $P^{(2)}$ are polynomials such that $|P^{(1)}(z_j)| + |P^{(2)}(z_j)| > 0$ solving (2.3), then the rational function $P^{(1)}/P^{(2)}$ is a solution of (2.1) with ω_j defined in (2.4).

In what follows we refer to interpolation problem (2.3) as the problem (I_n), and denote by $\mathbb{P}_\infty(I_n)$ the set of all its solutions. Some of the results described below for the polynomial vector-functions are taken from [10,11] where the interpolation problems also appears as well as its main objects like generators, etc.

2.1. The space of vector-polynomials

To describe the class of solutions of (2.3), we introduce the space

$$\mathbb{P}_\infty := \left\{ p(z) = \begin{pmatrix} P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix}, \quad P^{(i)},\ i = 1, 2,\ \text{are complex polynomials} \right\}.$$

This is a linear space with the standard operations. The zero element in this space is $0 = \binom{0}{0}$. We point out that $\mathbb{P}_\infty$ is also a module over the ring of polynomials:

$$Sp = \begin{pmatrix} SP^{(1)} \\ SP^{(2)} \end{pmatrix} \in \mathbb{P}_\infty$$

for any polynomial S.

Definition 2.1. The height of the vector-polynomial $p = \binom{P^{(1)}}{P^{(2)}} \neq 0$ is the number

$$h(p) := \begin{cases} 2\deg P^{(1)}, & \deg P^{(1)} > \deg P^{(2)}, \\ 2\deg P^{(2)} + 1, & \deg P^{(1)} \le \deg P^{(2)}. \end{cases} \tag{2.5}$$

As usual, $\deg 0 = -\infty$, so we put $h(0) := -\infty$.

It is obvious that

$$h(p) = \max\{2\deg P^{(1)}, 2\deg P^{(2)} + 1\} \tag{2.6}$$

and, for any polynomial S

$$h(Sp) = h(p) + 2\deg S. \tag{2.7}$$

The degrees of the components of the vector-polynomials can be written down in the following table:

height p	0	1	2	3	4	...	$2k$	$2k+1$	...
$\deg P^{(1)}$	$= 0$	≤ 0	$= 1$	≤ 1	$= 2$	...	$= k$	$\le k$	...
$\deg P^{(2)}$	$-\infty$	$= 0$	≤ 0	$= 1$	≤ 1	...	$\le k-1$	$= k$	...

The following proposition demonstrates that the notion of the height is a natural extension of the degree of polynomials.

Proposition 2.2. *If $h(p) \neq h(q)$, then $\forall a, b \in \mathbb{C}$*

$$h(ap + bq) = \max(h(p), h(q)).$$

If $h(p) = h(q) = n$, then
(1) $\forall\, a, b \in \mathbb{C}\ h(ap + bq) \le n$
(2) $\exists c \in \mathbb{C}$: $h(p + cq) \le n - 1$.

Proof. We prove (2), the rest is plain. If $n = 2k$, then $k = \deg P^{(1)} > \deg P^{(2)}$, $k = \deg Q^{(1)} > \deg Q^{(2)}$. So, there exists $c \in \mathbb{C}$ such that $\deg(P^{(1)} + c\,Q^{(1)}) \le k-1$. Also, since $\deg P^{(2)} \le k-1$

and $\deg Q^{(2)} \le k-1$, we see that $\deg(P^{(2)} + c\,Q^{(2)}) \le k-1$. Then, by (2.6)

$$h(p+cq) \le 2k-1 = n-1.$$

If $n = 2k+1$, then $k = \deg P^{(2)} \ge \deg P^{(1)}$, $k = \deg Q^{(2)} \ge \deg Q^{(1)}$. So, $\exists c \in \mathbb{C}$: $\deg(P^{(2)} + c\,Q^{(2)}) \le k-1$, $\deg(P^{(1)} + c\,Q^{(1)}) \le k$. Then, by (2.6)

$$h(p+cq) \le 2k = n-1. \quad \square$$

Consider the following basic system of vectors in $\mathbb{P}_\infty$:

$$e_{2k}(z) := \begin{pmatrix} z^k \\ 0 \end{pmatrix}, \qquad e_{2k+1}(z) := \begin{pmatrix} 0 \\ z^k \end{pmatrix}, \quad k \in \mathbb{Z}_+ = \{0, 1, \ldots\}. \tag{2.8}$$

It is clear that $h(e_n) = n$ for all n.

Proposition 2.3. *$\{e_n\}_{n\ge 0}$ is a basis in $\mathbb{P}_\infty$, i.e., for any $p \in \mathbb{P}_\infty$, $h(p) = m$, there exists unique $c_0, c_1, \ldots, c_m$, $c_m \ne 0$, such that*

$$p(z) = \sum_{k=0}^{m} c_k e_k(z)\,.$$

Proof. We use induction on m:

(1) The cases $m = 0, 1$ are checked immediately.
(2) Let $m = 2k+1$, then $k = \deg P^{(2)} \ge \deg P^{(1)}$, i.e., $P^{(2)}(z) = az^k + Q^{(2)}(z)$, $a \ne 0$, $\deg Q^{(2)}(z) \le k-1$. Then

$$q(z) := p(z) - ae_{2k+1}(z) = \begin{pmatrix} Q^{(1)}(z) \\ Q^{(2)}(z) \end{pmatrix},$$

where $Q^{(1)}(z) = P^{(1)}(z)$ and $\deg Q^{(1)} \le k$.

So, $h(q) \le 2k = m-1$, and we can apply the inductive hypothesis. The representation $q(z) = \sum_{k=0}^{m-1} c_k e_k(z)$ is unique, so the representation

$$p(z) = ae_m(z) + \sum_{k=0}^{m-1} c_k e_k(z)$$

is also unique.
(3) Let $m = 2k$. Then $P^{(1)}(z) = bz^k + Q^{(1)}(z)$, $b \ne 0$ and $\deg Q^{(1)} \le k-1$. Define $q(z) := p(z) - be_{2k} = \begin{pmatrix} Q^{(1)}(z) \\ Q^{(2)}(z) \end{pmatrix}$. Here $\deg Q^{(2)} \le k-1$, so $h(q) \le 2k-1 = m-1$ and we can again apply the inductive hypothesis. $\square$

The latter proposition can be extended in a natural way.

Proposition 2.4. *Let $\{g_n\}_{n\ge 0}$ be an arbitrary sequence of vector-polynomials, such that*

$$h(g_n) = n, \quad n \in \mathbb{Z}_+.$$

Then $\{g_n\}_{n\ge 0}$ is a basis in $\mathbb{P}_\infty$.

Proof. By Proposition 2.3

$$g_m(z) = \sum_{k=0}^{m} c_{m,k} e_k(z), \quad c_{m,m} \ne 0,$$

or in a vector-matrix form

$$\begin{pmatrix} g_0 \\ \vdots \\ g_m \end{pmatrix} = \begin{pmatrix} c_{00} & & & \\ c_{10} & c_{11} & & \\ \vdots & & \ddots & \\ c_{m0} & \dots & \dots & c_{mm} \end{pmatrix} \begin{pmatrix} e_0 \\ \vdots \\ e_m \end{pmatrix},$$

where the matrix of $(c_{m,k})$ is triangular. So,

$$\begin{pmatrix} e_0 \\ \vdots \\ e_m \end{pmatrix} = \begin{pmatrix} \widetilde{c}_{00} & & & \\ \widetilde{c}_{10} & \widetilde{c}_{11} & & \\ \vdots & & \ddots & \\ \widetilde{c}_{m0} & \dots & \dots & \widetilde{c}_{mm} \end{pmatrix} \begin{pmatrix} g_0 \\ \vdots \\ g_m \end{pmatrix}, \qquad \widetilde{c}_{m,m} \neq 0.$$

Since $\{e_n\}_{n\ge 0}$ is a basis, then so is $\{g_n\}_{n\ge 0}$. $\square$

2.2. Transforms in $\mathbb{P}_\infty$

A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines a transform of the vector-polynomials:

$$Ap(z) = A\begin{pmatrix} P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix} = \begin{pmatrix} aP^{(1)}(z) + bP^{(2)}(z) \\ cP^{(1)}(z) + dP^{(2)}(z) \end{pmatrix}.$$

Proposition 2.5. (1) *For an arbitrary matrix A and $p \in \mathbb{P}_\infty$*

$$h(Ap) \le h(p) + 1.$$

(2) *If A is upper-triangular (i.e., $c = 0$), then for arbitrary $p \in \mathbb{P}_\infty$*

$$h(Ap) \le h(p).$$

(3) *If A is lower-triangular (i.e., $b = 0$), then*

$$h(p) \le 2k+1 \Longrightarrow h(Ap) \le 2k+1.$$

Proof. (1) By (2.6)

$$\begin{aligned} &h(Ap) = \max\Big(2\deg(aP^{(1)} + bP^{(2)}), 2\deg(cP^{(1)} + \mathrm{d}P^{(2)}) + 1\Big), \\ &2\deg(aP^{(1)} + bP^{(2)}) \le 2\max(\deg P^{(1)}, \deg P^{(2)}) \\ &\qquad = \max(2\deg P^{(1)}, 2\deg P^{(2)}) \le h(p), \\ &2\deg(cP^{(1)} + dP^{(2)}) \le \max(2\deg P^{(1)}, 2\deg P^{(2)}) \le h(p), \\ &2\deg(cP^{(1)} + dP^{(2)}) + 1 \le h(p) + 1. \end{aligned}$$

(2) If $c = 0$, then $2\deg(dP^{(2)}) + 1 \le 2\deg P^{(2)} + 1 \le h(P)$.

(3) Let now $h(p) \le 2k+1$ and $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. Then

$$Ap(z) = \begin{pmatrix} aP^{(1)}(z) \\ cP^{(1)}(z) + dP^{(2)}(z) \end{pmatrix},$$

and we have by assumption $2\deg P^{(1)} \le 2k+1$, $2\deg P^{(2)} \le 2k$. So,

$$\deg(cP^{(1)} + dP^{(2)}) \le k \Longrightarrow h(Ap) \le 2k+1,$$

as claimed. □

Later on we will use the following property of the height, which is a simple consequence of (2.6): for $p = \binom{P^{(1)}}{P^{(2)}} \in \mathbb{P}_\infty$ we have

$$\begin{cases} h(p) \le 2k+1 \Longrightarrow h\begin{pmatrix} (z-a)P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix} \le 2k+2, \\ h(p) \le 2k \Longrightarrow h\begin{pmatrix} P^{(1)}(z) \\ (z-a)P^{(2)}(z) \end{pmatrix} \le 2k+1. \end{cases} \tag{2.9}$$

2.3. The generators of interpolation problem

It is clear that solutions of (2.3) form a module over the ring of polynomials in $\mathbb{P}_\infty$, i.e., if r and q are solutions of (2.3), then so is $Sr+Tq$ for arbitrary polynomials S and T. The goal of this subsection is to show that this module has exactly two generators, and to study their properties.

Recall that $\mathbb{P}_\infty(I_n)$ is the set of all solutions of (2.3). Set

$$h(I_n) := \min\{h(q) :\ q \in \mathbb{P}_\infty(I_n), \quad q \ne 0\},$$

which we call the height of the interpolation problem.

Definition 2.6. We say that $r \in \mathbb{P}_\infty(I_n)$ is a minimal generator of (2.3), if

$$h(r) = h(I_n).$$

Proposition 2.7. *The minimal generator of (I_n) is unique up to a constant factor.*

Proof. Let r_1 and r_2 be two minimal generators. By Proposition 2.2, $\exists a \in \mathbb{C}$: $h(ar_1 + r_2) \le h(I_n) - 1$. But $ar_1 + br_2$ is a solution of (2.3). Since r_1 and r_2 are minimal nontrivial solutions of (2.3), we conclude $ar_1 + br_2 = 0$. □

A trivial (nonzero) solution of (2.3) $P^{(1)} = P^{(2)} = \prod_j (z - z_j)$ provides the bound $h(I_n) \le 2n+1$. It turns out that this bound can be improved immensely.

Theorem 2.8. $h(I_n) \le n$.

Proof. The following non-negative matrices of rank 1 play a key role in our consideration:

$$\sigma_k = \begin{pmatrix} |\alpha_k^{(1)}|^2 & \bar\alpha_k^{(1)}\alpha_k^{(2)} \\ \bar\alpha_k^{(2)}\alpha_k^{(1)} & |\alpha_k^{(2)}|^2 \end{pmatrix} = \begin{pmatrix} \bar\alpha_k^{(1)} \\ \bar\alpha_k^{(2)} \end{pmatrix} (\alpha_k^{(1)},\ \alpha_k^{(2)}), \quad k = 1, \ldots, n.$$

It is clear that

$$|\alpha_k^{(1)} P^{(1)}(z_k) + \alpha_k^{(2)} P^{(2)}(z_k)|^2 = p^*(z_k)\sigma_k\, p(z_k),$$

$$p^*(z) := (\overline{P^{(1)}(z)},\ \overline{P^{(2)}(z)}), \qquad p(z) := \begin{pmatrix} P^{(1)}(z) \\ P^{(2)}(z) \end{pmatrix},$$

so problem (2.3) is equivalent to

$$p^*(z_k)\sigma_k\, p(z_k) = 0, \quad k = 1, \ldots, n. \tag{2.10}$$

We proceed by induction on n.

1. For $n = 1$ we have a nontrivial solution

$$p := \begin{pmatrix} -\alpha_1^{(2)} \\ \alpha_1^{(1)} \end{pmatrix}, \quad h(p) \le 1.$$

2. Suppose that we have already proved the result for n, and we want to prove it for $n+1$. The forthcoming construction depends on whether n is odd or even.

Let $n = 2k+1$, and consider the problem (I_{n+1}) (2.10) with $n+1$ data. If $\alpha_j^{(1)} = 0$ for all $j = 1, \ldots, n+1$, then the vector-polynomial $p = \binom{1}{0}$, $h(p) = 0$, is a solution of (I_n), and we are done. So, suppose without loss of generality that $\alpha_{n+1}^{(1)} \ne 0$ (otherwise enumerate the points $z_1, \ldots, z_{n+1}$). The upper-triangular matrix

$$\Omega_0 := \begin{pmatrix} (\alpha_{n+1}^{(1)})^{-1} & -\alpha_{n+1}^{(2)}(\alpha_{n+1}^{(1)})^{-1} \\ 0 & 1 \end{pmatrix},$$

satisfies

$$\Omega_0^*\, \sigma_{n+1}\, \Omega_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.11}$$

and

$$\Omega_0^*\, \sigma_j\, \Omega_0 = \begin{pmatrix} |\beta_j^{(1)}|^2 & \bar\beta_j^{(1)}\beta_j^{(2)} \\ \bar\beta_j^{(2)}\beta_j^{(1)} & |\beta_j^{(2)}|^2 \end{pmatrix}; \quad j = 1, \ldots, n, \tag{2.12}$$

with some numbers $\beta_j^{(1)}, \beta_j^{(2)}$, $j = 1, \ldots, n$. Put

$$\gamma_j^{(1)} := (z_{n+1} - z_j)\beta_j^{(1)}; \qquad \gamma_j^{(2)} := \beta_j^{(2)}, \quad j = 1, \ldots, n,$$

and consider an auxiliary interpolation problem $(\widetilde{I}_n)$:

$$\gamma_j^{(1)} P^{(1)}(z_j) + \gamma_j^{(2)} P^{(2)}(z_j) = 0, \qquad j = 1, \ldots, n.$$

By the induction hypothesis, there exists a solution $q \in \mathbb{P}_\infty$, $h(q) \le n = 2k+1$, of this problem:

$$q^*(z_j)\widetilde{\sigma}_j\, q(z_j) = 0, \quad j = 1, \ldots, n,$$

where

$$\widetilde{\sigma}_j = \begin{pmatrix} |\gamma_j^{(1)}|^2 & \bar\gamma_j^{(1)}\gamma_j^{(2)} \\ \bar\gamma_j^{(2)}\gamma_j^{(1)} & |\gamma_j^{(2)}|^2 \end{pmatrix} = \begin{pmatrix} \overline{z_{n+1} - z_j} & 0 \\ 0 & 1 \end{pmatrix} \Omega_0^*\sigma_j\Omega_0 \begin{pmatrix} z_{n+1} - z_j & 0 \\ 0 & 1 \end{pmatrix}.$$

Define a vector-polynomial

$$r(z) = \begin{pmatrix} R^{(1)}(z) \\ R^{(2)}(z) \end{pmatrix} := \Omega_0 \begin{pmatrix} z_{n+1} - z & 0 \\ 0 & 1 \end{pmatrix} q(z). \tag{2.13}$$

For r we have

$$r^*(z_j)\,\sigma_j\, r(z_j) = q^*(z_j)\,\widetilde{\sigma}_j\, q(z_j) = 0, \quad j = 1, \ldots, n,$$

and for $j = n + 1$

$$r^*(z_{n+1})\,\sigma_{n+1}\,r(z_{n+1}) = q^*(z_{n+1})\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}q(z_{n+1}) = 0,$$

by (2.11). So, r is a solution of (I_{n+1}).

Since $h(q) \le 2k + 1$, then by the upper inequality in (2.9) we have

$$h\begin{pmatrix}(z_{n+1} - z)Q^{(1)}(z)\\ Q^{(2)}(z)\end{pmatrix} \le 2k + 2 = n + 1,$$

so, by Proposition 2.5, $h(r) \le n + 1$, as needed.

Let $n = 2k$, and (I_{n+1}) (2.10) be the interpolation problem with $n + 1$ data. If $\alpha_j^{(2)} = 0$ for all $j = 1, \ldots, n + 1$, then the vector-polynomial $p = \binom{0}{1}$, $h(p) = 1$, is a solution of (I_n), and we are done. So, suppose as above, that $\alpha_{n+1}^{(2)} \ne 0$. The lower-triangular matrix

$$\Omega_1 := \begin{pmatrix}1 & 0\\ -\alpha_{n+1}^{(1)}(\alpha_{n+1}^{(2)})^{-1} & (\alpha_{n+1}^{(2)})^{-1}\end{pmatrix},$$

satisfies

$$\Omega_1^*\,\sigma_{n+1}\,\Omega_1 = \begin{pmatrix}0&0\\0&1\end{pmatrix}, \tag{2.14}$$

and

$$\Omega_1^*\,\sigma_j\,\Omega_1 = \begin{pmatrix}|\delta_j^{(1)}|^2 & \bar\delta_j^{(1)}\delta_j^{(2)}\\ \bar\delta_j^{(2)}\delta_j^{(1)} & |\delta_j^{(2)}|^2\end{pmatrix}; \quad j = 1, \ldots, n. \tag{2.15}$$

Put

$$\lambda_j^{(2)} := (z_{n+1} - z_j)\delta_j^{(2)}; \qquad \lambda_j^{(1)} := \delta_j^{(1)}, \quad j = 1, \ldots, n,$$

and consider an auxiliary interpolation problem $(\hat I_n)$:

$$\lambda_j^{(1)}P^{(1)}(z_j) + \lambda_j^{(2)}P^{(2)}(z_j) = 0, \quad j = 1, \ldots, n,$$

By the induction hypothesis, there exists a solution $q \in \mathbb{P}_\infty$, $h(q) \le n = 2k$, of this problem:

$$q^*(z_j)\widetilde\sigma_j\,q(z_j) = 0, \quad j = 1, \ldots, n,$$

where

$$\widetilde\sigma_j = \begin{pmatrix}|\lambda_j^{(1)}|^2 & \bar\lambda_j^{(1)}\lambda_j^{(2)}\\ \bar\lambda_j^{(2)}\lambda_j^{(1)} & |\lambda_j^{(2)}|^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \overline{z_{n+1} - z_j}\end{pmatrix}\Omega_1^*\sigma_j\Omega_1\begin{pmatrix}1 & 0\\ 0 & z_{n+1} - z_j\end{pmatrix}.$$

Define a vector-polynomial

$$r(z) = \begin{pmatrix}R^{(1)}(z)\\ R^{(2)}(z)\end{pmatrix} := \Omega_1\begin{pmatrix}1 & 0\\ 0 & z_{n+1} - z\end{pmatrix}q(z). \tag{2.16}$$

For r we have

$$r^*(z_j)\,\sigma_j\,r(z_j) = q^*(z_j)\,\widetilde\sigma_j\,q(z_j) = 0, \quad j = 1, \ldots, n,$$

and for $j = n + 1$

$$r^*(z_{n+1})\,\sigma_{n+1}\,r(z_{n+1}) = q^*(z_{n+1}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} q(z_{n+1}) = 0,$$

by (2.14). So, r is a solution of (I_{n+1}).

Since $h(q) \le 2k$, then by the lower inequality in (2.9) we have

$$h\begin{pmatrix} Q^{(1)}(z) \\ (z_{n+1} - z)Q^{(2)}(z) \end{pmatrix} \le 2k + 1 = n + 1,$$

so, by part (3), Proposition 2.5 (Ω_1 is lower-triangular), $h(r) \le 2k + 1 = n + 1$. The proof is complete. □

If r is a minimal generator of (I_n), then $Sr \in \mathbb{P}_\infty(I_n)$ for any polynomial S. So, the question arises naturally whether $\mathbb{P}_\infty(I_n) = \{Sr\}$, S a polynomial. The answer is negative: it turns out that the module of solutions of (I_n) has exactly one more generator. Denote $\mathbb{P}'_\infty(I_n) = \mathbb{P}_\infty(I_n) \setminus \{Sr\}$. It is shown in Theorem 2.10 below that this set is nonempty, so the following definition makes sense.

Definition 2.9. We say that $q \in \mathbb{P}_\infty(I_n)$ is a second generator of (I_n), if

$$h(q) = \min\{h(p),\ p \in \mathbb{P}'_\infty(I_n)\}.$$

Theorem 2.10. *The set $\mathbb{P}'_\infty(I_n)$ is nonempty. Furthermore, the height of any second generator is $h(q) = 2n + 1 - h(I_n)$.*

Proof. We show that there is a solution of (I_n) of the height $\le 2n + 1 - h(I_n)$, which is not of the form Sr.

Let $h(I_n) = k \le n$. Pick arbitrary different numbers $z_{n+1}, z_{n+2}, \ldots, z_{2n+1-k} \in \mathbb{C}$ distinct from $z_1, \ldots, z_n$. Take numbers $\alpha_j^{(1)}, \alpha_j^{(2)}$, $j = n + 1, n + 2, \ldots, 2n + 1 - k$ in such a way that

$$\alpha_j^{(1)} R^{(1)}(z_j) + \alpha_j^{(2)} R^{(2)}(z_j) \ne 0, \quad j = n + 1, n + 2, \ldots, 2n + 1 - k, \tag{2.17}$$

where $r = \begin{pmatrix} R^{(1)} \\ R^{(2)} \end{pmatrix}$ is the minimal generator of (I_n). Consider the interpolation problem

$$\alpha_j^{(1)} P^{(1)}(z_j) + \alpha_j^{(2)} P^{(2)}(z_j) = 0, \quad j = 1, 2, \ldots, 2n + 1 - k. \tag{2.18}$$

By Theorem 2.8, there exists a nonzero vector-polynomial $p = \begin{pmatrix} P^{(1)} \\ P^{(2)} \end{pmatrix} \ne 0$, which solves this problem, and $h(p) \le 2n + 1 - k$.

Suppose that $p = Sr$ for a polynomial S. Then, by (2.18),

$$S(z_j)\left[\alpha_j^{(1)} R^{(1)}(z_j) + \alpha_j^{(2)} R^{(2)}(z_j)\right] = 0, \quad j = n + 1, n + 2, \ldots, 2n + 1 - k,$$

so by (2.17),

$$S(z_{n+1}) = S(z_{n+2}) = \cdots = S(z_{2n+1-k}) = 0.$$

Since $S(z) \not\equiv 0$, we see that $\deg S(z) \ge n + 1 - k$, and by (2.9)

$$h(p) = h(Sr) = h(r) + 2\deg S \ge h(r) + 2(n + 1 - k) = 2n + 2 - k,$$

which leads to contradiction with $h(p) \le 2n+1-k$. So $p \in \mathbb{P}'_\infty(I_n)$ and $h(p) \le 2n+1-k$, as claimed.

Let now $q = \binom{Q^{(1)}}{Q^{(2)}}$ be any second generator, so $h(q) \le 2n+1-k$. We prove next $h(q) \ge 2n+1-k$.

We have

$$\begin{cases} \alpha_j^{(1)} R^{(1)}(z_j) + \alpha_j^{(2)} R^{(2)}(z_j) = 0, \\ \alpha_j^{(1)} Q^{(1)}(z_j) + \alpha_j^{(2)} Q^{(2)}(z_j) = 0, \end{cases} \qquad j = 1, \ldots, n,$$

so

$$\det \begin{pmatrix} R^{(1)}(z_j) & R^{(2)}(z_j) \\ Q^{(1)}(z_j) & Q^{(2)}(z_j) \end{pmatrix} = R^{(1)}(z_j) Q^{(2)}(z_j) - R^{(2)}(z_j) Q^{(1)}(z_j) = 0, \tag{2.19}$$

$j = 1, \ldots, n$. Suppose that $h(q) \le 2n - k$, which implies by (2.6)

$$\deg Q^{(1)} \le n - \frac{k}{2}, \qquad \deg Q^{(2)} \le n - \frac{k}{2} - 1.$$

If $h(r) = k$ is even, then

$$\deg R^{(1)} = \frac{k}{2}, \qquad \deg R^{(2)} \le \frac{k}{2} - 2,$$

and so

$$\deg \left(R^{(1)} Q^{(2)} - R^{(2)} Q^{(1)} \right) \le n - 1.$$

If k is odd, then

$$\deg R^{(2)} = \frac{k-1}{2}, \qquad \deg R^{(1)} \le \frac{k-1}{2},$$

and again

$$\deg \left(R^{(1)} Q^{(2)} - R^{(2)} Q^{(1)} \right) \le n - 1.$$

It follows now from (2.19) that

$$R^{(1)}(z) Q^{(2)}(z) - R^{(2)}(z) Q^{(1)}(z) \equiv 0. \tag{2.20}$$

Let T be the greatest common divisor of $R^{(1)}$ and $R^{(2)}$, so (2.20) turns into $X^{(1)} T Q^{(2)} = X^{(2)} T Q^{(1)}$, with relatively prime $X^{(1)} := R^{(1)}/T$ and $X^{(2)} := R^{(2)}/T$. Hence $S^{(1)} := Q^{(1)}/X^{(1)}$ is a polynomial, and so is $S^{(2)} := Q^{(2)}/X^{(2)}$. Now $X^{(1)} T Q^{(2)} = X^{(2)} T Q^{(1)}$ implies $S^{(1)} = S^{(2)} = S$ and

$$Q^{(i)} = \frac{R^{(i)}}{T} \cdot S, \quad i = 1, 2. \tag{2.21}$$

Note that all roots of T are among the nodes of interpolation. Indeed, if $T(w) = 0$ and $w \notin \{z_1, \ldots, z_n\}$, then

$$\widehat{r}(z) = \begin{pmatrix} \widehat{R}^{(1)} \\ \widehat{R}^{(2)} \end{pmatrix} = \frac{1}{z - w} \begin{pmatrix} R^{(1)} \\ R^{(2)} \end{pmatrix} \in \mathbb{P}_\infty(I_n)$$

and $h(\widehat{r}) < h(r)$, which is impossible since r is a minimal generator of (I_n). Hence, $w = z_l$.

It remains only to show that $S(z_l) = 0$. Assume that $S(z_l) \neq 0$. Then

$$\alpha_l^{(1)} Q^{(1)}(z_l) + \alpha_l^{(2)} Q^{(2)}(z_l) = 0$$

and (2.21) imply

$$\lim_{z \to z_l} \left\{ \alpha_l^{(1)} \frac{R^{(1)}(z)}{T(z)} + \alpha_l^{(2)} \frac{R^{(2)}(z)}{T(z)} \right\} = 0.$$

So

$$\lim_{z \to z_l} \left\{ \alpha_l^{(1)} \frac{R^{(1)}(z)}{(z - z_l)^{n_l}} + \alpha_l^{(2)} \frac{R^{(2)}(z)}{(z - z_l)^{n_l}} \right\} = 0,$$

where $T = (z - z_l)^{n_l} \widetilde{T}$, $\widetilde{T}(z_l) \neq 0$. Since $R^{(i)} = (z - z_l)^{n_l} \widetilde{R}^{(i)}$, $i = 1, 2$, we see that

$$\widetilde{r}(z) = \frac{1}{(z - z_l)^{n_l}}\, r(z) \in \mathbb{P}_\infty(I_n)$$

and clearly $h(\widetilde{r}) < h(r)$, which again leads to contradiction with r being the minimal generator.

Finally, since all the roots of T are among $\{z_1, \dots, z_k\}$ and $S(z_1) = \cdots = S(z_k) = 0$, $P = S/T$ is a polynomial, and by (2.21) $Q^{(i)} = PR^{(i)}$, which contradicts to $q \in \mathbb{P}'_\infty(I_n)$. The proof is complete. □

Remark. In fact we have proven that each solution $q \in \mathbb{P}'_\infty(I_n)$ with $h(q) \leq 2n + 1 - h(I_n)$ is a second generator. It is also not hard to see that each solution $q \in \mathbb{P}(I_n)$ with $h(q) = 2n+1-h(I_n)$ is a second generator.

Theorem 2.11. *Each solution of the problem (I_n) has the form*

$$p(z) = S(z)r(z) + T(z)q(z), \tag{2.22}$$

where r and q are the minimal and second generators of (I_n), respectively, and S and T are polynomials. Conversely, each vector-polynomial of the form (2.22) with arbitrary polynomials S and T belongs to $\mathbb{P}_\infty(I_n)$.

Constructive algorithms of finding q and r are available.

Proof. We only prove the first statement. Let $h(I_n) = k$. Consider a system of vector-polynomials $\{f_j\}_{j \geq 0}$ defined as follows:

$$\begin{aligned} &f_k = r, \qquad f_{2n+1-k} = q, \\ &f_j = e_j, \quad j = 0, 1, \dots, k-1; \qquad f_{k+2j-1} = e_{k+2j-1}, \quad j = 1, 2, \dots, n-k, \\ &f_{k+2j} = z^j r, \qquad f_{2n+1-k+2j} = z^j q; \quad j \in \mathbb{N}, \end{aligned}$$

e_i are in (2.8). It is easy to check that $h(f_j) = j$ for all j, so by Proposition 2.4 this system is a basis in $\mathbb{P}_\infty$, and in particular each $p \in \mathbb{P}_\infty(I_n)$ admits a unique representation in the form

$$p(z) = \sum_{i=0}^{k-1} a_i e_i + \sum_{i=0}^{n-1-k} b_i e_{k+1+2i} + S_1(z)r(z) + T_1(z)q(z).$$

Since $p, r, q \in \mathbb{P}_\infty(I_n)$ then $\sum_{i=0}^{k-1} a_i e_i + \sum_{i=0}^{n-1-k} b_i e_{k+1+2i}$ is also solution of (I_n). But its height is less then $2n+1-k$, so, according to the definition of a second generator,

$$\sum_{i=0}^{k-1} a_i e_i + \sum_{i=0}^{n-1-k} b_i e_{k+1+2i} = \widetilde{S}(z) r(z).$$

Hence p is of the form (2.22) with $S = S_1 + \widetilde{S}$, $T = T_1$.

It remains to explain why the above algorithms are constructive. The inductive proof of Theorem 2.8 gives us a constructive algorithm of obtaining a solution of the problem (I_n), whose height is at most n. Thus, in view of Theorem 2.11, the obtained solution has the form $S(z)r(z)$, where $S(z)$ is a polynomial. By using the Euclid algorithm we can find the greatest common divisor of the entries of this vector-polynomial, so after reduction we come to the minimal generator in a constructive way starting from the interpolation data.

Also, the proof of Theorem 2.10 gives us a constructive algorithm of finding a solution of (I_n), whose height is exactly $2n+1-k$. Thus, the obtained solution is exactly a second generator. Therefore, we have constructive algorithm of finding both the minimal and the second generator of the problem (I_n). $\square$

Thus, the following theorem holds for interpolation problem (2.1):

Theorem 2.12. *Each solution of problem (2.1) has the form*

$$\frac{S(z)R^{(1)}(z) + T(z)Q^{(1)}(z)}{S(z)R^{(2)}(z) + T(z)Q^{(2)}(z)}, \tag{2.23}$$

where $r = \binom{R^{(1)}}{R^{(2)}}$ and $q = \binom{Q^{(1)}}{Q^{(2)}}$ are minimal and second generators of (I_n), and S and T are polynomials. Conversely, if r and q are the minimal and second generators of (I_n), and S and T are such polynomials that the numerator and the denominator in (2.23) have no common roots, then (2.23) is a solution of (2.1).

Remark. Roughly speaking, the task of giving *a* description for the set of the solutions of rational interpolation problem (2.1), is obvious. Let $\frac{R^{(1)}}{R^{(2)}}$ be *any* rational function solving problem (2.1). Then all the functions of the type $\frac{R^{(1)}}{R^{(2)}} + \frac{Q^{(1)}}{Q^{(2)}}$, where $\frac{Q^{(1)}}{Q^{(2)}}$ is an arbitrary rational function vanishing in the nodes of interpolation, will be all the solutions of the rational interpolation problem. However, such a description does not permit us to predict the degrees of the numerator and denominator as well as other properties needed in applications. So, we cannot obtain in this way a rational function which solves the interpolation problem and has the prescribed properties. For example, in the interpolation problem appearing in the next section we will need a rational function with monic numerator and denominator of concrete degrees, such that the free term of the numerator equals 1. Thus, more elaborated results are required. Certainly, we do not think that the two descriptions for the solutions of the interpolation problem, mentioned above, are the only possible.

3. Reduction of MISP to rational interpolation

For the definitions, notations and basic properties of finite CMV matrices see, for example, [14,9,6]. We will add some more to the list.

Let $\mathcal{C} = \mathcal{C}(\alpha_0, \dots, \alpha_{n-2}; \beta)$ be a finite CMV matrix with Verblunsky's parameters $(\alpha_0, \dots, \alpha_{n-2}; \beta)$ and the system of the monic Szegő polynomials $\{\Phi_0, \dots, \Phi_{n-1}; \widetilde{\Phi}_n\}$. They satisfy the *Szegő recurrence relations*

$$\begin{aligned} &\Phi_k(z) = z\Phi_{k-1}(z) - \bar{\alpha}_{k-1}\Phi^*_{k-1}(z), \quad k = 1, 2, \dots, n-1, \ \Phi_0 \equiv 1, \\ &\widetilde{\Phi}_n(z) = z\Phi_{n-1}(z) - \bar{\beta}\Phi^*_{n-1}(z). \end{aligned} \tag{3.1}$$

As is known,

$$\widetilde{\Phi}_n(z) = \prod_{j=1}^{n}(z - \zeta_j), \qquad \Sigma(\mathcal{C}) = \{\zeta_j\}_1^n$$

a spectrum of $\mathcal{C}$, $\zeta_j \neq \zeta_i$, $j \neq i$. Put

$$\kappa_m = \prod_{j=0}^{m-1}(1 - |\alpha_j|^2)^{-1/2}, \qquad m = 0, 1, \dots, n-1; \qquad \kappa_0 = 1, \tag{3.2}$$

and for appropriate values of the indices define

$$x_{2k}(z) := z^{-k}\kappa_{2k}\Phi_{2k}(z), \qquad x_{2k+1}(z) := z^{-k-1}\kappa_{2k+1}\Phi^*_{2k+1}(z), \tag{3.3}$$

where $\Phi^*_{2k+1} = z^{-2k-1}\overline{\Phi_{2k+1}(1/\bar{z})}$.

For each eigenvalue ζ_j the following equality

$$\mathcal{C}X_j = \zeta_j X_j, \qquad X_j = [x_0(\zeta_j), \dots, x_{n-1}(\zeta_j)]^{\mathrm{t}}, \tag{3.4}$$

gives (along with (3.3)) an explicit expression for the eigenvectors of $\mathcal{C}$ in terms of the Szegő polynomials and Verblunsky parameters. (3.4) is proved in [12, Lemma 4.3.14], for infinite CMV matrices. For finite matrices the argument is similar. As a matter of fact, the following more precise result holds.

Proposition 3.1. *For $z \in \mathbb{C} \setminus \{0\}$ and $X(z) = [x_0(z), \dots, x_{n-1}(z)]^{\mathrm{t}}$ the equality holds*

$$(z - \mathcal{C})X = z^{-[n/2]}\kappa_{n-1}\widetilde{\Phi}_n(z)v_n, \quad v_n \in \mathbb{C}^n, \ \|v_n\| = 1.$$

Due to the sieving procedure which provides a simple relation between measures with the Verblunsky coefficients $(\alpha_0, \alpha_1, \dots, \alpha_{n-2}; \beta)$ and $(0, \alpha_0, 0, \alpha_1, \dots, 0, \alpha_{n-2}, 0; \beta)$ (see, e.g., [12, Example 1.6.14]) we will assume without loss of generality that n is an even number: $n = 2l$. Throughout the rest of the paper we assume that the last Verblunsky coefficient is known, and for simplicity put $\beta = 1$ (the general case can be reduced to this particular one by appropriate rotation of the measure, (see [12, Section 3.2]). The $\mathcal{LM}$ factorization takes the form $\mathcal{C}(\alpha_0, \dots, \alpha_{2l-2}; 1) = \mathcal{LM}$ with

$$\mathcal{L} = \begin{pmatrix} \theta(\alpha_0) & & \\ & \ddots & \\ & & \theta(\alpha_{2l-2}) \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} 1 & & & & \\ & \theta(\alpha_1) & & & \\ & & \ddots & & \\ & & & \theta(\alpha_{2l-3}) & \\ & & & & 1 \end{pmatrix}, \tag{3.5}$$

$$\Theta(\alpha_j) = \begin{pmatrix} \bar{\alpha_j} & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \quad |\alpha_j| < 1, \ \rho_j = \sqrt{1 - |\alpha_j|^2} > 0.$$

Put

$$U = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} & J \\ \mathcal{O} & \cdots & J & \mathcal{O} \\ \cdots & \cdots & \cdots & \cdots \\ J & \mathcal{O} & \cdots & \mathcal{O} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.6}$$

the orthogonal $2l \times 2l$ matrix, and consider the *reflection* of $\mathcal{C}$

$$\mathcal{C}_r := U\mathcal{C}U = U\mathcal{L}U \cdot U\mathcal{M}U = \mathcal{L}_r \cdot \mathcal{M}_r. \tag{3.7}$$

It is clear from (3.5)–(3.7), that

$$\begin{aligned} \mathcal{L}_r &= \begin{pmatrix} \theta(-\bar{\alpha}_{2l-2}) & & \\ & \ddots & \\ & & \theta(-\bar{\alpha}_0) \end{pmatrix}; \\ \mathcal{M}_r &= \begin{pmatrix} 1 & & & & \\ & \theta(-\bar{\alpha}_{2l-3}) & & & \\ & & \ddots & & \\ & & & \theta(-\bar{\alpha}_1) & \\ & & & & 1 \end{pmatrix}, \end{aligned} \tag{3.8}$$

so

$$\mathcal{C}_r = \mathcal{C}(\lambda_0, \ldots, \lambda_{2l-2}; 1), \quad \lambda_k := -\bar{\alpha}_{2l-2-k}, \quad k = 0, 1, \ldots, 2l-2, \tag{3.9}$$

is also a CMV matrix corresponding to the "reversed" Verblunsky parameters. We denote the Szegő polynomials for $\mathcal{C}_r$ by $\{\Lambda_0, \ldots, \Lambda_{n-1}; \widetilde{\Lambda}_n\}$.

Obviously, $\Sigma(\mathcal{C}_r) = \Sigma(C) = \{\zeta_j\}_1^n$ and

$$\mathcal{C}_r Y_j = \zeta_j Y_j, \quad Y_j = [y_0(\zeta_j), \ldots, y_{n-1}(\zeta_j)]^{\mathrm{t}},$$

where y_k are in (3.3) for the matrix $\mathcal{C}_r$. On the other hand, by (3.4) and (3.7)

$$\mathcal{C}_r \hat{X}_j = \zeta_j \hat{X}_j, \quad \hat{X}_j = [x_{n-1}(\zeta_j), \ldots, x_0(\zeta_j)]^{\mathrm{t}}.$$

Since the spectrum is simple, the vectors Y_j and $\hat{X}_j$ are proportional:

$$Y_j = c_j \hat{X}_j, \qquad y_{k-1}(\zeta_j) = c_j x_{n-k}(\zeta_j); \quad k, j = 1, 2, \ldots, n,$$

or

$$\frac{y_{k-1}(\zeta_j)}{y_k(\zeta_j)} = \frac{x_{n-k}(\zeta_j)}{x_{n-k-1}(\zeta_j)}, \quad k = 1, 2, \ldots, n-1. \tag{3.10}$$

Under the *mixed inverse spectral problem* (MISP) we mean the reconstruction of a CMV matrix $\mathcal{C} = \mathcal{C}(\alpha_0, \ldots, \alpha_{n-2}; 1)$, or equivalently, of a set of Verblunsky parameters $\alpha_0, \ldots, \alpha_{n-2} \in \mathbb{D}$, when a part $\{\zeta_j\}_{j=1}^m$ of its spectrum and a part of the system $(\alpha_0, \ldots, \alpha_{n-2})$ are known.

Here is the simplest problem of this type. Assume that we know $(\alpha_0, \ldots, \alpha_{n-3})$ as well as two eigenvalues $\zeta_1 \neq \zeta_2$, and α_{n-2} is to be found so that $\zeta_{1,2} \in \Sigma(\mathcal{C})$. Once Φ_{n-2} is known, we apply the Szegő recurrences to obtain

$$\widetilde{\Phi}_n(z) = z(z + \alpha_{n-2})\,\Phi_{n-2}(z) - (z\bar{\alpha}_{n-2} + 1)\,\Phi^*_{n-2},$$

so

$$b(\zeta_j) = \tau_j, \qquad b(\lambda) = \frac{\lambda + \alpha_{n-2}}{1 + \lambda\bar{\alpha}_{n-2}}, \quad j = 1, 2, \tag{3.11}$$

$$\tau_j = \frac{\Phi^*_{n-2}(\zeta_j)}{\zeta_j \Phi_{n-2}(\zeta_j)}, \quad j = 1, 2.$$

The question is whether α_{n-2} is uniquely determined from the interpolation problem (3.11). An elementary analysis of (3.11) shows that it has a unique solution as long as $\tau_1\zeta_1 \neq \tau_2\zeta_2$, that is,

$$\frac{\Phi^*_{n-2}(\zeta_1)}{\Phi_{n-2}(\zeta_1)} \neq \frac{\Phi^*_{n-2}(\zeta_2)}{\Phi_{n-2}(\zeta_2)}, \tag{3.12}$$

it has infinitely many solutions if $\tau_2 = -\zeta_1$ and $\tau_1 = -\zeta_2$, or

$$\frac{\Phi^*_{n-2}(\zeta_1)}{\Phi_{n-2}(\zeta_1)} = \frac{\Phi^*_{n-2}(\zeta_2)}{\Phi_{n-2}(\zeta_2)} = -\zeta_1\zeta_2,$$

and it has no solutions at all, if $\tau_1\zeta_1 = \tau_2\zeta_2$, but $\tau_1 \neq -\zeta_2$, $\tau_2 \neq -\zeta_1$. It is not hard to check that each situation may occur for interpolation problem (3.11). However, if the existence of CMV matrix $\mathcal{C}$ with $\zeta_{1,2} \in \Sigma(\mathcal{C})$ is supposed, the existence of the solution of problem (3.11) is guaranteed and the problem of finding α_{n-2} may have either unique or infinitely many solutions (see Example 1 below).

Since the Blaschke product Φ_{n-2}/Φ^*_{n-2} of order $n-2$ cannot take the same value on the n-point set $\Sigma(\mathcal{C})$, there always exists such a pair $\zeta_1 \neq \zeta_2$ in $\Sigma(\mathcal{C})$, that (3.12) holds, so α_{n-2} is uniquely determined.

The general MISP for CMV matrices we study here looks as follows. Let $n = 2l$ be even. Given first $n-m-1$ Verblunsky parameters $\alpha_0, \ldots, \alpha_{n-m-2}$, and $2m$ eigenvalues $\zeta_1, \ldots, \zeta_{2m}$, $1 \le m \le n/2 = l$, find the rest m parameters $\alpha_{n-m-1}, \ldots, \alpha_{n-2}$ and thereby restore the whole matrix $\mathcal{C}$.[1] Our main result provides the conditions for this problem to have a unique solution.

Consider a pair of CMV matrices with the "known" parameters $\mathcal{C}(\alpha_0, \ldots, \alpha_{n-m-3}; 1)$ and $\mathcal{C}(\alpha_0, \ldots, \alpha_{n-m-2}; 1)$ and the systems of the monic Szegő polynomials

$$\{\Phi_0, \ldots, \Phi_{n-m-2}; \widetilde{\Phi}_{n-m-1}\}, \qquad \{\Phi_0, \ldots, \Phi_{n-m-1}; \widetilde{\Phi}_{n-m}\},$$

respectively. By the Szegő recurrences (3.1)

$$\begin{aligned}\widetilde{\Phi}_{n-m-1}(z) &= z\Phi_{n-m-2}(z) - \Phi^*_{n-m-2}(z),\\ \Phi_{n-m-1}(z) &= z\Phi_{n-m-2}(z) - \bar{\alpha}_{n-m-2}\Phi^*_{n-m-2}(z),\end{aligned}$$

so

$$\Phi_{n-m-1}(z) - \widetilde{\Phi}_{n-m-1}(z) = (1 - \bar{\alpha}_{n-m-2})\Phi^*_{n-m-2}(z). \tag{3.13}$$

Similarly, for the pair $\mathcal{C}(\lambda_0, \ldots, \lambda_{m-1}; 1)$ and $\mathcal{C}(\lambda_0, \ldots, \lambda_m; 1)$ of "unknown" CMV matrices, λ_j from (3.9) with the Szegő polynomials $\{\Lambda_0, \ldots, \Lambda_m; \widetilde{\Lambda}_{m+1}\}$ and $\{\Lambda_0, \ldots, \Lambda_{m+1}; \widetilde{\Lambda}_{m+2}\}$, respectively, one has

$$\Lambda_{m+1}(z) - \widetilde{\Lambda}_{m+1}(z) = (1 - \bar{\lambda}_m)\Lambda^*_m(z). \tag{3.14}$$

[1] $2m$ "real" parameters are given to find m "complex" ones.

Now write (3.10) with $k = m + 1$:

$$\frac{y_m(\zeta_j)}{y_{m+1}(\zeta_j)} = \frac{x_{n-m-1}(\zeta_j)}{x_{n-m-2}(\zeta_j)}, \quad j = 1, \dots, n, \tag{3.15}$$

and observe that the right-hand side of (3.15) is known for $j = 1, 2, \dots, 2m$. Indeed, let, e.g., m be odd (for even m the calculation is the same). Then by (3.3)

$$x_{n-m-2}(z) = z^{-\frac{n-m-1}{2}} \kappa_{n-m-2} \Phi^*_{n-m-2}(z); \qquad x_{n-m-1}(z) = z^{-\frac{n-m-1}{2}} \kappa_{n-m-1} \Phi_{n-m-1}(z),$$

so in view of (3.13) and (3.2)

$$\begin{aligned}\frac{x_{n-m-1}(\zeta_j)}{x_{n-m-2}(\zeta_j)} &= \frac{\kappa_{n-m-1}}{\kappa_{n-m-2}} \frac{\Phi_{n-m-1}(\zeta_j)}{\Phi^*_{n-m-2}(\zeta_j)} \\ &= (1 - |\alpha_{n-m-2}|^2)^{-1/2} \frac{\widetilde{\Phi}_{n-m-1}(\zeta_j) + (1 - \bar{\alpha}_{n-m-2}) \Phi^*_{n-m-2}(\zeta_j)}{\Phi^*_{n-m-2}(\zeta_j)} \\ &= \rho^{-1}_{n-m-2} \left\{ \frac{\widetilde{\Phi}_{n-m-1}(\zeta_j)}{\Phi^*_{n-m-2}(\zeta_j)} + 1 - \bar{\alpha}_{n-m-2}, \right\}, \quad \rho_i = (1 - |\alpha_i|^2)^{1/2}.\end{aligned}$$

In the same way

$$y_m = z^{-\frac{m+1}{2}} \kappa_{m,r} \Lambda^*_m(z), \qquad y_{m+1} = z^{-\frac{m+1}{2}} \kappa_{m+1,r} \Lambda_{m+1}(z),$$

and with

$$\kappa_{m,r} = \prod_{j=0}^{m-1} (1 - |\lambda_j|^2)^{-1/2} = \prod_{j=n-m-1}^{n-2} (1 - |\alpha_j|^2)^{-1/2}$$

we have

$$\begin{aligned}\frac{y_m(\zeta_j)}{y_{m+1}(\zeta_j)} &= \frac{\kappa_{m,r}}{\kappa_{m+1,r}} \cdot \frac{\Lambda^*_m(\zeta_j)}{\Lambda_{m+1}(\zeta_j)} = \rho_{m,r} \left\{ \frac{\widetilde{\Lambda}_{m+1}(\zeta_j) + (1 - \bar{\lambda}_m) \Lambda^*_m(\zeta_j)}{\Lambda^*_m(\zeta_j)} \right\}^{-1} \\ &= \rho_{m,r} \left\{ \frac{\widetilde{\Lambda}_{m+1}(\zeta_j)}{\Lambda^*_m(\zeta_j)} + 1 + \alpha_{n-m-2} \right\}^{-1}, \\ &\quad \rho_{m,r} = (1 - |\lambda_m|^2)^{-1/2} = \rho_{n-m-2}.\end{aligned}$$

Using (3.15), we end up with the following equalities for $j = 1, 2, \dots, 2m$

$$\frac{\widetilde{\Lambda}_{m+1}(\zeta_j)}{\Lambda^*_m(\zeta_j)} = -1 - \alpha_{n-m-2} + \frac{1 - |\alpha_{n-m-2}|^2}{\frac{\widetilde{\Phi}_{n-m-1}(\zeta_j)}{\Phi^*_{n-m-2}(\zeta_j)} + 1 - \bar{\alpha}_{n-m-2}}. \tag{3.16}$$

As the last step, we express the ratios in terms of the Weyl functions w and w_r of the known $\mathcal{C}(\alpha_0, \dots, \alpha_{n-m-3}; 1)$ and unknown $\mathcal{C}(\lambda, \dots, \lambda_{m-1}; 1)$, respectively (see [12,6]). The Weyl function of a CMV matrix $\mathcal{C}(\alpha_0, \dots, \alpha_k; \beta)$ is defined by

$$w(\mathcal{C}, z) = \int_{\mathbb{T}} \frac{\mathrm{d}\sigma(t)}{t - z} = \frac{\Phi_{k-1}(z)}{\widetilde{\Phi}_k(z)},$$

where σ is the spectral measure of $\mathcal{C}$. So,

$$w(z) = \frac{\Phi_{n-m-2}(z)}{\widetilde{\Phi}_{n-m-1}(z)}, \qquad w_r(z) = \frac{\Lambda_m(z)}{\widetilde{\Lambda}_{m+1}(z)}.$$

Indeed, for $|z| = 1$

$$\widetilde{\Phi}_{n-m-1}(z) = \prod_{i=1}^{n-m-1} (z - z_i), \quad |z_i| = 1,$$

$$\widetilde{\Phi}_{n-m-1}(0) = (-1)^{n-m-1} \prod_{i=1}^{n-m} z_i = -\bar{\beta} = -1,$$

$$\overline{\widetilde{\Phi}_{n-m-1}(z)} = \prod_{i=1}^{n-m-1} (z^{-1} - z_i^{-1}) = -z^{-n+m+1} \widetilde{\Phi}_{n-m-1}(z),$$

so $\overline{\widetilde{\Phi}_{n-m-1}(\zeta_j)} = -\zeta_j^{-n+m+1} \widetilde{\Phi}_{n-m-1}(\zeta_j)$,

$$\frac{\widetilde{\Phi}_{n-m-1}(\zeta_j)}{\Phi^*_{n-m-2}(\zeta_j)} = -\frac{\zeta_j^{n-m-1} \overline{\widetilde{\Phi}_{n-m-1}(\zeta_j)}}{\zeta_j^{n-m-2} \overline{\Phi_{n-m-2}(\zeta_j)}} = -\zeta_j \overline{W(\zeta_j)}, \quad W := \frac{1}{w},$$

and, similarly,

$$\frac{\widetilde{\Lambda}_{m+1}(\zeta_j)}{\Lambda^*_m(\zeta_j)} = -\zeta_j \overline{W_r(\zeta_j)}, \quad W_r := \frac{1}{w_r}.$$

Finally, after (3.16) and the above considerations, we come to the following interpolation problem for the Weyl function of the "unknown" CMV matrix $\mathcal{C}(\lambda_0, \ldots, \lambda_{m-1}; 1)$

$$W_r(\zeta_j) = \zeta_j \frac{(1 + \bar{\alpha}_{n-m-2}) \left\{1 - \alpha_{n-m-2} - \bar{\zeta}_j W(\zeta_j)\right\} - (1 - |\alpha_{n-m-2}|^2)}{\left\{1 - \alpha_{n-m-2} - \bar{\zeta}_j W(\zeta_j)\right\}} =: \omega_j, \quad (3.17)$$

$j = 1, 2, \ldots, 2m$, or

$$P^{(1)}(\zeta_j) - \omega_j P^{(2)}(\zeta_j) = 0, \quad j = 1, \ldots, 2m, \quad (3.18)$$

with ω_j defined in (3.17), which we have denoted by (I_{2m}) in the previous section. Now $\omega_j \neq \infty$ since all zeros of Λ_m are in the open unit disk $\mathbb{D}$. The above argument shows that (3.18) has a nontrivial solution[2]

$$\lambda = \begin{pmatrix} \Lambda^{(1)} \\ \Lambda^{(2)} \end{pmatrix} = \begin{pmatrix} \widetilde{\Lambda}_{m+1} \\ \Lambda_m \end{pmatrix}$$

and

$$h(\lambda) = 2m + 2. \quad (3.19)$$

Proposition 3.2. *For problem* (3.18) $h(I_{2m}) \geq 2m - 1$.

[2] Again, we assume that $\mathcal{C} = \mathcal{C}(\alpha_0, \ldots, \alpha_{n-2}; 1)$ with given $\alpha_0, \ldots, \alpha_{n-m-2}$ and the eigenvalues $\zeta_1, \ldots, \zeta_{2m}$ does exist.

Proof. Let r be the minimal generator of (3.18), and suppose that $h(r) \leq 2m - 2$. By Theorem 2.10 for the second generator q one has $h(q) \geq 2m + 3$. It follows now from Theorem 2.11 and (3.19) that $\lambda = Sr$ with $\deg S \geq 2$, so

$$\widetilde{\Lambda}_{m+1}(z) = S(z)R^{(1)}(z), \qquad \Lambda_m(z) = S(z)R^{(2)}(z),$$

which is impossible, for $\widetilde{\Lambda}_{m+1}$ and Λ_m have no common zeros. $\square$

There is some more information available about the solution λ. Specifically,

$$\deg \widetilde{\Lambda}_{m+1} = m + 1, \qquad \deg \Lambda_m = m \tag{3.20}$$

and

$$\widetilde{\Lambda}_{m+1}(0) = -1. \tag{3.21}$$

In view of Theorem 2.8, it is easy to conclude from Proposition 3.2 that if the data of interpolation problem (3.18) corresponds to a CMV matrix, then either $h(I_{2m}) = 2m$ or $h(I_{2m}) = 2m - 1$.

Theorem 3.3. *Let for the minimal generator r of problem (3.18) $R^{(1)}(0) \neq 0$ holds. Then (3.18) has a unique solution and, hence, the solution of the MISP is unique. The constructive algorithm of finding $\mathcal{C}$ from the mixed data is available.*

Proof. By Theorem 2.11

$$\begin{aligned} \Lambda^{(1)}(z) &= S(z)R^{(1)}(z) + T(z)Q^{(1)}(z), \\ \Lambda^{(2)}(z) &= S(z)R^{(2)}(z) + T(z)Q^{(2)}(z), \end{aligned}$$

where r and q are the minimal and second generators for problem (3.18), respectively. Proposition 3.2 reads that either $h(r) = 2m$ or $h(r) = 2m - 1$, so by Theorem 2.10 either $h(q) = 2m + 1$ or $h(q) = 2m + 2$. In the first case

$$\deg R^{(1)} = m, \qquad \deg R^{(2)} \leq m - 1, \qquad \deg Q^{(1)} \leq m, \qquad \deg Q^{(2)} = m,$$

and in the second one

$$\deg R^{(1)} \leq m - 1, \qquad \deg R^{(2)} = m - 1, \qquad \deg Q^{(1)} = m + 1, \qquad \deg Q^{(2)} \leq m.$$

In view of the degrees of $\Lambda^{(1)}$ and $\Lambda^{(2)}$, in both cases

$$\deg S(z) = 1, \qquad \deg T(z) = 0,$$

i.e., for $\Lambda^{(j)}$ we have

$$\begin{aligned} \Lambda^{(1)}(z) &= (az + b)R^{(1)}(z) + cQ^{(1)}(z), \\ \Lambda^{(2)}(z) &= (az + b)R^{(2)}(z) + cQ^{(2)}(z). \end{aligned} \tag{3.22}$$

For fixed r and q, it is easy to see that a and c are uniquely determined by condition (3.20). If $R^{(1)}(0) \neq 0$, then b is also uniquely determined by (3.21), but if $Q^{(1)}(0) = 0$, the problem has either no solutions, or infinitely many solutions. But since the data of the problem is taken from the CMV matrix, the solution does exist. So, if $R^{(1)}(0) = 0$, the interpolation problem has infinitely many solutions and, hence, the MISP may have infinitely many solutions. $\square$

In view of this theorem, two natural questions arise.

(1) Is it possible for the minimal generator to have $R^{(1)}(0) = 0$ when (3.17) is related to MISP for a certain CMV matrix?

(2) Is it possible for the MISP to have more than one solution if $R^{(1)}(0) = 0$?

Both answers are positive, so in some special cases MISP with nonunique solutions does exist, although the number of the "pieces of information" in the inverse data is equal to the number of parameters to reconstruct. We provide examples for both possible cases $h(I_{2m}) = 2m - 1$ (Example 1) and $h(I_{2m}) = 2m$ (Example 2).

Example 1. Let $-1 < b < 1$ and $\mathcal{C} = \mathcal{C}(0, 0, b; 1)$. By the Szegő recurrences

$$\Phi_1(z) = z, \qquad \Phi_2 = z^2, \qquad \Phi_3 = z^3 - b, \qquad \Phi_3^*(z) = -bz^3 + 1,$$

and

$$\widetilde{\Phi}_4(z) = z\Phi_3(z) - \Phi_3^*(z) = (z^2 - 1)(z^2 + bz + 1),$$

so the eigenvalues are

$$\zeta_{1,2} = \pm 1, \qquad \zeta_{3,4} = -\frac{b}{2} \pm \mathrm{i}\sqrt{1 - \frac{b^2}{4}}.$$

We see that the pair ζ_1, ζ_2 does not determine b uniquely, although any other pair does.

However, the MISP of (nonunique) reconstruction of b by the two eigenvalues $\zeta_{1,2} = \pm 1$ is still possible. Find the right-hand side of (3.17) for this case. First, consider the Weyl function of the "known" left matrix $\mathcal{C}(0; 1)$. Its Szegő polynomials are $\Phi_1(z) = z$ and $\widetilde{\Phi}_2(z) = z\Phi_1 - 1 \cdot \Phi_1^* = z^2 - 1$. So, the reciprocal of its Weyl function is $W(z) = \frac{z^2-1}{z}$ and the right-hand side of (3.17) is

$$\omega_j = \zeta_j \cdot \frac{(1+0)\{1 - 0 - \bar{\zeta}_j W(\zeta_j)\} - (1-0)}{1 - 0 - \bar{\zeta}_j W(\zeta_j)} = \zeta_j \cdot \frac{-\bar{\zeta}_j W(\zeta_j)}{1 - \bar{\zeta}_j W(\zeta_j)}.$$

For $\zeta_j = \pm 1$ we have $\omega_j = 0$, so, according to (3.17), the reciprocal of the Weyl function of the matrix $\mathcal{C}_r = \mathcal{C}(-b; 1)$ satisfies

$$W_r(\pm 1) = 0.$$

Remind that it also must satisfy additional conditions (3.20) and (3.21) with $m = 2$.

So, following the procedure of solving the MISP, described above, to reconstruct the inverse of the Weyl function of $\mathcal{C}(-b; 1)$, we need to reconstruct a rational function $\frac{P^{(1)}}{P^{(2)}}$ such that

$$\begin{aligned} &\frac{P^{(1)}(z)}{P^{(2)}(z)}\Big|_{z=\pm 1} = 0, \qquad \deg P^{(1)} = 2, \qquad \deg P^{(2)}(z) = 1, \\ &P^{(1)}, P^{(2)} \text{ are monic and } P^{(1)}(0) = 1. \end{aligned} \tag{3.23}$$

The corresponding interpolation problem for the vector-functions is

$$P^{(1)}(\pm 1) = 0. \tag{3.24}$$

According to Proposition 3.2, the minimal generator of the problem (3.24) must have the height ≥ 1, and according to Theorem 2.8, it must have the height ≤ 2. In fact, it can be immediately checked that the nontrivial vector-function of minimal height, corresponding to this problem, is

$\binom{0}{1}$, whose height is 1. Further, since the height of the minimal generator is 1 and there are 2 points of interpolation, according to Theorem 2.10, the second generator must have the height 4 and there is no solutions of height 2. In fact, it is evident that the vector-polynomial $\binom{(z+1)(z-1)}{0}$, whose height is 4, solves (3.24), and there is no solutions of height 2. Finally, the general solution of (3.23), is

$$\frac{P^{(1)}(z)}{P^{(2)}(z)} = \frac{1\cdot(z+1)(z-1)+(z+a)\cdot 0}{1\cdot 0+(z+b)\cdot 1} = \frac{(z+1)(z-1)}{z+b},$$

with arbitrary number b (cf. (3.22)). However, only those solutions with additional condition $|b| < 1$ give us not only a solution of (3.23), but also the Weyl function of a CMV matrix of the type $\mathcal{C}(-b; 1)$. In fact, let us find directly the Weyl function of $\mathcal{C}(-b; 1)$. Its Szegő polynomials are:

$$\begin{cases} \Lambda_1(z) = z+b; \\ \widetilde{\Lambda}_2(z) = z\Lambda_1(z) - \Lambda_1^*(z) = z(z+b) - (bz+1) = (z+1)(z-1). \end{cases}$$

So,

$$W_r(z) = \frac{\widetilde{\Lambda}_2}{\Lambda_1} = \frac{(z+1)(z-1)}{z+b},$$

as was to be checked.

Example 2. Consider a family of CMV matrices of order 4: $\mathcal{C}(0, -y, -x; 1)$; $-1 < x, y < 1$, and analyze the MISP of reconstruction of the unknown x, y by the four eigenvalues. Calculate for them the Szegő polynomials:

$$\begin{aligned}
&\Phi_1(z) = z, \qquad \Phi_1^*(z) = 1; \\
&\Phi_2 = z\Phi_1 + y\Phi_1^* = z^2 + y, \qquad \Phi_2^* = 1 + z^2 y; \\
&\Phi_3 = z\Phi_2 + x\Phi_2^* = z(z^2+y) + x(1+z^2y) = z^3 + xyz^2 + yz + x, \\
&\Phi_3^* = 1 + xyz + yz^2 + xz^3; \\
&\widetilde{\Phi}_4(z) = z\Phi_3 - 1\cdot\Phi_3^* = z(z^3 + xyz^2 + yz + x) - 1 - xyz - yz^2 - xz^3 \\
&\qquad = (z^4 - 1) + (xy - x)z(z^2 - 1) = (z^2-1)(z^2 + (xy-x)z + 1).
\end{aligned}$$

Introducing the notation

$$k := xy - x, \tag{3.25}$$

we express the eigenvalues of $\mathcal{C}$ as

$$\Sigma: \quad \zeta_{1,2} = \pm 1; \qquad \zeta_{3,4} = -\frac{k}{2} \pm \mathrm{i}\sqrt{1 - \frac{k^2}{4}}; \quad \zeta_3 \neq \zeta_4 \text{ and } \zeta_4 = \bar{\zeta}_3. \tag{3.26}$$

Hence, if x and y are related by (3.25) and k is fixed, we have an infinite family of CMV matrices $\mathcal{C}(x, y, 0; 1)$ with the same spectrum Σ (3.26). According to the general theory the auxiliary Weyl functions $W_r(z)$ and $W(z)$ of the matrices $\mathcal{C}_r = \mathcal{C}(x, y; 1)$ and $\mathcal{C}(1)$, resp., take the same values on Σ for different x and y, related by (3.25). Find the right-hand side of (3.17) for this case. First, consider the Weyl function of the "known" left matrix $\mathcal{C}(1)$. Its

Szegő polynomials are $\Phi_0(z) = 1$ and $\widetilde{\Phi}_1(z) = z\Phi_0 - 1 \cdot \Phi_0^* = z - 1$. So, the inverse of its Weyl function is $W(z) = z - 1$ and the right-hand side of (3.17) is

$$\omega_j = \zeta_j \cdot \frac{(1+0)\{1 - 0 - \bar{\zeta}_j W(\zeta_j)\} - (1-0)}{1 - 0 - \bar{\zeta}_j W(z_j)} = \zeta_j \cdot \frac{-\bar{\zeta}_j(\zeta_j - 1)}{1 - \bar{\zeta}_j(\zeta_j - 1)} = \zeta_j(1 - \zeta_j).$$

Let us directly check that the left-hand side of (3.17) coincide with the obtained numbers. The Szegő polynomials of $\mathcal{C}_r = \mathcal{C}(x, y; 1)$ are

$$\begin{aligned}
&\Lambda_1(z) = z - x, \qquad \Lambda_1^*(z) = 1 - xz;\\
&\Lambda_2(z) = z\Lambda_1 - y\Lambda_1^* = z^2 + (xy - x)z - y, \qquad \Lambda_2^*(z) = -yz^2 + (xy - x)z + 1;\\
&\widetilde{\Lambda}_3(z) = z\Lambda_2 - \Lambda_2^* = z^3 + (xy - x)z^2 - yz + yz^2 - (xy - x)z - 1\\
&\qquad = (z - 1)(z^2 + (k + y + 1)z + 1);\\
&W_r(z) = \frac{\widetilde{\Lambda}_3(z)}{\Lambda_2(z)} = \frac{(z - 1)(z^2 + (k + y + 1)z + 1)}{z^2 + kz - y};\\
&W_r(1) = 0 = \omega_1;\\
&W_r(-1) = \frac{-2(2 - (k + y + 1))}{1 - k - y} = -2 = \omega_2;
\end{aligned}$$

and, since $\zeta_j^2 + k\zeta_j + 1 = 0$; $j = 3, 4$, we have

$$W_r(\zeta_j) = \frac{(\zeta_j - 1)(y + 1)\zeta_j}{-(y + 1)} = \zeta_j(1 - \zeta_j) = \omega_j; \quad j = 3, 4.$$

If we solved the MISP following the procedure described above, we have to find (nonuniquely) the inverse Weyl function $W_r(z)$ from its value in the four eigenvalues and the additional conditions for the numerator and denominator. In this example we will restrict ourselves by illustrating that the minimal generator of the corresponding interpolation problem does not satisfy the conditions of Theorem 3.3, which actually cause the existence of infinitely many solutions.

Consider the interpolation problem, corresponding to this case:

$$\begin{cases} R^{(1)}(\zeta_j) - \omega_j R^{(2)}(\zeta_j); \quad j = 1, 2, 3, 4; \zeta_j \in \Sigma;\\ \omega_1 = 0, \quad \omega_2 = -2, \quad \omega_{3,4} = \zeta_{3,4}(1 - \zeta_{3,4}).\end{cases}$$

(since $\zeta_j \neq 1$, we have $\omega_j \neq 0$). We are looking for the solution of height 4: $R^{(1)} = z^2 + \alpha z + \beta$, $R^{(2)} = \gamma z + \delta$, so

$$\begin{cases} 1 + \alpha + \beta = 0;\\ 1 - \alpha + \beta + 2(-\gamma + \delta) = 0;\\ \zeta_j^2 + \alpha\zeta_j + \beta - \omega_j(\gamma\zeta_j + \delta) = 0, \quad j = 3, 4.\end{cases}$$

Since $\zeta_j^2 + k\zeta_j + 1 = 0$, the last 2 equations can be rewritten as

$$\begin{aligned}
&(\alpha - k)\zeta_j + \beta - 1 - \omega_j(\gamma\zeta_j + \delta) = 0,\\
&(\alpha - k)\zeta_j - \alpha - 2 - \omega_j(\gamma\zeta_j + \delta) = 0; \quad j = 3, 4.
\end{aligned}$$

Exclude from these equations first γ, then δ:

$$(1)\ \begin{cases} (\alpha - k)\zeta_3\omega_4 - (\alpha + 2)\omega_4 - \gamma\zeta_3\omega_3\omega_4 - \delta\omega_3\omega_4 = 0;\\ (\alpha - k)\zeta_4\omega_3 - (\alpha + 2)\omega_3 - \gamma\zeta_4\omega_4\omega_3 - \delta\omega_3\omega_4 = 0.\end{cases} \Rightarrow$$

$$(\alpha-k)(\zeta_3\omega_4-\zeta_4\omega_3)-(\alpha+2)(\omega_4-\omega_3)-\gamma|\omega_3|^2(\zeta_3-\zeta_4)=0.$$

$$\begin{aligned}
&(\alpha-k)(\zeta_3-\zeta_4)-(\alpha+2)(\omega_4-\omega_3)=\gamma|1-\zeta_3|^2(\zeta_3-\zeta_4);\\
&\gamma=\frac{\alpha-k}{|1-\zeta_3|^2}+(a+2)\frac{\omega_3-\omega_4}{(\zeta_3-\zeta_4)|1-\zeta_3|^2}=\frac{\alpha-k}{|1-\zeta_3|^2}\\
&\qquad+(\alpha+2)\frac{(1+k)}{|1-\zeta_3|^2}=\frac{(\alpha+1)(k+2)}{|1-\zeta_3|^2}.
\end{aligned}$$

But $|1-\zeta_3|^2=2-\zeta_3-\bar{\zeta}_3=2+k\Rightarrow\gamma=\alpha+1$.

$$(2)\ \begin{cases}(\alpha-k)\zeta_3\cdot\zeta_4\omega_4-(\alpha+2)\zeta_4\omega_4-\gamma\zeta_3\omega_3\zeta_4\omega_4-\delta\omega_3\cdot\zeta_4\omega_4=0;\\(\alpha-k)\zeta_4\cdot\zeta_3\omega_3-(\alpha+2)\zeta_3\omega_3-\gamma\zeta_4\omega_3\zeta_4\omega_4-\delta\omega_4\cdot\zeta_3\omega_4=0.\end{cases}\Rightarrow$$

$$\begin{aligned}
&(\alpha-k)(\omega_4-\omega_3)-(\alpha+2)(\zeta_4\omega_4-\zeta_3\omega_3)-\delta|1-\zeta_3|^2(\zeta_4-\zeta_3)=0,\\
&\zeta_4\omega_4-\zeta_3\omega_3=\zeta_4^2(1-\zeta_4)-\zeta_3^2(1-\zeta_3)\\
&\qquad=(\zeta_4-\zeta_3)(\zeta_4+\zeta_3-\zeta_4^2-\zeta_3^2-1)=(\zeta_4-\zeta_3)(-k^2-k+1),\\
&\omega_4-\omega_3=(\zeta_4-\zeta_3)(1-\zeta_3-\zeta_4)=(\zeta_4-\zeta_3)(1+k).
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\alpha-k)(1+k)+(\alpha+2)(k^2+k-1)=\delta(2+k);\\
&\alpha k(k+2)+(k-1)(k+2)=\delta(k+2)\Rightarrow\delta=\alpha k+k-1.
\end{aligned}$$

(3) We have from the second equation $\alpha=\delta-\gamma\Rightarrow\alpha=\alpha k+k-2\Rightarrow$

$$\begin{aligned}
&\alpha=-1;\qquad\beta=0;\qquad\gamma=0;\qquad\delta=-1,\\
&R^{(1)}(z)=z^2-z;\qquad R^{(2)}(z)=-1.
\end{aligned}$$

It can be immediately checked that the solution of height 3 (such that $R^{(1)}=\alpha z+\beta$, $R^{(2)}=z+\gamma$) does not exist, so $r=\binom{R^{(1)}}{R^{(2)}}$ is the minimal solution. As we see, $R^{(1)}(0)=0$.

Finally, rewriting

$$W_r(z)=\frac{(z-1)(z^2+(k+1)z+1)+(z^2-z)\cdot y}{z^2+kz+(-1)\cdot y};$$

we see in the nondetermined (arbitrary) up to constant factor terms of the numerator $(z^2-z)\cdot y$ and denominator $-1\cdot y$, the components of the minimal generator (cf. (3.22) with $b=y$).

Remark. Assume that $2m+1$ eigenvalues $\zeta_1,\ldots,\zeta_{2m},\zeta_{1m+1}$ are known. Since

$$w_2(z)=\frac{\Psi_m(z)}{\widetilde{\Psi}_{m+1}(z)}=\frac{z^m+\cdots}{z^{m+1}+\cdots}$$

the interpolation problem $w_2(\zeta_j)=\Omega_j$; $j=1,2,\ldots,2m+1$ has obviously the unique solution.

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